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The Youla Parameterization for Nonlinear Feedback Systems with Additive Disturbances

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Abstract

Building on the work presented in [9, 10, 11, 12], a construction of the Youla Parameterization for nonlinear feedback systems is presented in which the feedback loop is disturbed by additive disturbances. The construction of the Youla parameterization of [9, 10] may then be shown to be stable and well-posed in the presence of external disturbances.

1. Introduction

In linear systems theory the Youla parameterization of the class of all controllers stabilizing a given plant has proven to be a useful tool in the analysis and design of linear feedback systems, see for example [19] and the references contained therein.

A number of authors have tried to extend these results to nonlinear systems, for example Hammer [3, 4, 5], Chen [1, 2], Moore and co-workers [7, 6] and Verma [16, 17, 18]. None of these works proved a fully satisfactory nonlinear version of the linear results, in that either state space expressions were not available, or the Youla parameterization could not be fully derived.

To date the most successful results in this direction have been derived by Paice and van der Schaft [9] where the Youla parameterization is derived via the use of *stable kernel representations.* These results had the restriction of having to assume that the feedback system was undisturbed by external influences, but the advantages of having explicit state space formulations of the input-output results, and allowing all operators involved to be nonlinear.

In this paper we show how these results may be extended to give a parameterization of the class of feedback pairs which are stable with respect to additive disturbances at the input and output, as in Figure 1.

2. Preliminaries

In this section we introduce the definitions necessary for our results, and quote some previous results.

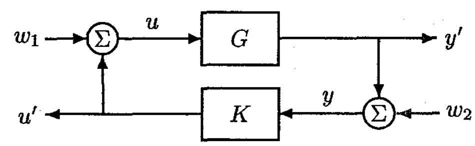

Figure 1: The feedback system $\{G, K\}$.

2.1 Signal Spaces and Stability

Throughout the paper a *signal space*, $\mathcal{Z}$ is taken to be a vector space of functions from a given time domain to a Euclidean vector space. The signal space is partitioned into two disjoint subsets, the stable signals, $\mathcal{Z}^s$, and the unstable signals, $\mathcal{Z}^u$. In this paper it is assumed that $\mathcal{Z}^s$ is a vector space, $0 \in \mathcal{Z}^s$, thus the sum of any two stable signals is stable, while the sum of a stable signal with an unstable signal is unstable.

An operator $\Sigma : \mathcal{U} \to \mathcal{Y}$ is said to be *stable* when for all $u \in \mathcal{U}^s$, $\Sigma u \in \mathcal{Y}^s$. This is also known as *bounded input, bounded output (BIBO)* stability.

An invertible operator is called *unimodular* when it is stable and has a stable inverse.

2.2 The Feedback System $\{G, K\}$

We shall be interested in the analysis of the feedback system consisting of the plant $G : \mathcal{U} \to \mathcal{Y}$ and controller $K : \mathcal{Y} \to \mathcal{U}$, $\mathcal{Y}$ and $\mathcal{U}$ signal spaces, as depicted in Figure 1. In particular, the existence and stability of the internal signals given the disturbances needs to be determined. This leads to the following definitions of well-posedness and stability.

Definition 2.1 *The system* $\{G, K\}$ *is* well-posed *iff for all inputs* w_1, w_2 *the outputs* u, u', y *and* y' *exist and are uniquely determined,* i.e. *iff the closed-loop system input-output operator from* w_1, w_2 *to* u, y, *namely*

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \textit{exists.} \tag{1}$$

Definition 2.2 *The system* $\{G, K\}$, *assumed well-*

posed, is said to be internally stable *iff for all stable inputs* w_1, w_2 *the outputs* u, u', y *and* y' *are stable. This is equivalent to*

$$\begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \text{ is stable.} \qquad (2)$$

2.3 Right Coprime Factorizations

The well-posedness and stability of the feedback loop may be characterized by the existence and stability of an operator constructed from the *right coprime factorizations (rcfs)* of the plant and controller. Right coprimeness is defined as follows.

Definition 2.3 *Let* M, N *be a stable operators such that for* $G : \mathcal{U} \to \mathcal{Y}$

$$G = NM^{-1},\ N : \mathcal{Z}_K \to \mathcal{Y},\ M : \mathcal{Z}_K \to \mathcal{U} \qquad (3)$$

Then M, N *is a* right coprime factorization *of* G *iff for all unbounded inputs* $s \in \mathcal{Z}_K$, Ms *or* Ns *is unbounded.*

Remark 2.4 A sufficient condition for the coprimeness of M, N is that there exists a stable operator $L : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_K$ such that

$$L \begin{bmatrix} M \\ N \end{bmatrix} = I_{\mathcal{Z}_K}$$

where $I_{\mathcal{Z}_K}$ is the identity operator on $\mathcal{Z}_K$. See [8] for details.

Given that K has a *rcf*

$$K = UV^{-1},\ U : \mathcal{Z}_G \to \mathcal{U},\ V : \mathcal{Z}_G \to \mathcal{Y} \qquad (4)$$

the well-posedness and stability of $\{G, K\}$ may be characterized.

Theorem 2.5
Given $\{G, K\}$, *and* $G = NM^{-1}$ *and* $K = UV^{-1}$ rcfs, *then* $\{G, K\}$ *is well-posed iff*

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \text{ exists} \qquad (5)$$

and is internally stable iff

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \text{ is stable} \qquad (6)$$

The existence and stability of this matrix was used in [11] to construct *stable kernel representations*, a nonlinear generalization of stable left factorizations which may be used in the construction of the Youla parameterization. These concepts are reviewed next.

2.4 Stable Kernel Representations

Stable kernel representations (*skrs*) were introduced in [9] as generalized left factorizations, see also [11]. Definitions were introduced for well-posedness and stability of feedback systems within such a framework which are not equivalent to the definitions above. Thus, within this paper we shall rename these definitions *null-well-posedness* and *null-stability*, as they correspond to the case that the external inputs to the system are zero. To simplify the notation, we shall not consider the general case where initial conditions must always be notated. Instead we assume that the operators used here are well defined input-output operators, that is the initial conditions for state space realizations of the operators are known and fixed throughout.

A *kernel representation* of the operator $\Sigma : \mathcal{U} \to \mathcal{Y}$ is an operator $R_\Sigma : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_\Sigma$ such that

$$y = \Sigma u \quad \text{iff} \quad R_\Sigma(u, y) = 0 \qquad (7)$$

The results derived in the sequel depend on considering the case $R_\Sigma(u, y) = z \neq 0$. It is often useful to know that each z characterizes a new input-output operator. That R_Σ has this property is known as *well-definedness.*

Definition 2.6 *A kernel representation* R_Σ *of* Σ *is said to be* well-defined *if for each* $z \in \mathcal{Z}$ *the map* $\Sigma_z : \mathcal{U} \to \mathcal{Y}$ *exists, so that for all* $u \in \mathcal{U}$, $y = \Sigma_z u$ *iff* $R_\Sigma(u, y) = z$.

This is equivalent to the existence of a pseudo-inverse $R_\Sigma^\# : \mathcal{U} \times \mathcal{Z} \to \mathcal{Y}$ *satisfying*

$$\begin{aligned} R_\Sigma^\#(u, R_\Sigma(u, y)) &= y \\ R_\Sigma(u, R_\Sigma^\#(u, z)) &= z \end{aligned}$$

In particular we shall be interested in *skrs* which are coprime, in analogy with the linear case.

Definition 2.7 *A kernel representation* R_Σ *of* Σ *is said to be coprime if it is stable and has a stable right inverse. That is, there exists a stable operator* $T : \mathcal{Z} \to \mathcal{Y} \times \mathcal{U}$ *such that*

$$RT = Z : \mathcal{Z} \mapsto \mathcal{Z},\ \text{unimodular} \qquad (8)$$

We now consider the case when $G : \mathcal{U} \to \mathcal{Y}$ and $K : \mathcal{Y} \to \mathcal{U}$ have kernel representations, and are connected in a feedback loop. In particular we are interested in the case that

$$\begin{aligned} R_G(u, y) &= z_G \\ R_K(y, u) &= z_K \end{aligned} \qquad (9)$$

has a solution for u and y. When the solutions exist and are unique, the system is called *null-well-posed.* When the solutions are stable for stable z_G, z_K the system is said to be *null-stable.* For simplicity we define the closed-loop or system kernel operator, as follows.

$$\begin{aligned} R_{GK} &: \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_K \times \mathcal{Z}_G \\ R_{GK}(u,y) &= \begin{pmatrix} R_K(y,u) \\ R_G(u,y) \end{pmatrix} \end{aligned} \tag{10}$$

Additionally we introduce the following notation

$$z_{GK} = \begin{pmatrix} z_K \\ z_G \end{pmatrix}, \qquad \mathcal{Z}_{GK} = \mathcal{Z}_K \times \mathcal{Z}_G$$

Definition 2.8 *The system* $\{G, K\}$ *is* null-well-posed *iff for all* $z_{GK} \in \mathcal{Z}_{GK}$, *the solution* (u, y) *to (10) is unique. That is,*

$$R^{-1}_{\{G,K\}} : \mathcal{Z}_{GK} \to \mathcal{U} \times \mathcal{Y} \qquad \textit{exists.} \tag{11}$$

Definition 2.9 *The closed loop system* $\{G, K\}$ *with* skr R_{GK} *as in (10) is* mull-stable *if it is null-well-posed, and for all* $z_{GK} \in \mathcal{Z}^s_{GK}$ *the solution* (y, u) *to (10) is stable, iff* $(z_{GK}, x_{GK}) \in \mathcal{B}_{GK}$. *That is,*

$$R^{-1}_{\{G,K\}} : \mathcal{Z}_{GK} \to \mathcal{U} \times \mathcal{Y} \qquad \textit{is stable.} \tag{12}$$

Remark 2.10 In [11] it was shown that if the closed loop system is null-stable, then the kernel representations are stable and coprime, and that there exist right coprime factorizations for the plant and controller. Well-posedness and stability of the feedback loop may then be checked by applying Theorem 2.5.

The dual result relating the construction of *skrs* given right coprime factorizations of the plant and controller in a stable, well-posed feedback loop is more important here.

Theorem 2.11 [11]
Let $\{G, K\}$ *be a well-posed and stable feedback system and suppose that* G *and* K *have right coprime factorizations, as in (3), (4). Then, by Theorem 2.5,*

$$\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \qquad \textit{exists and is stable.}$$

Define the functions $R_K : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_K$ *and* $R_G : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_G$ *by the following equations*

$$\begin{bmatrix} R_K(y+N(0),\ -u+M(0)) \\ R_G(u+U(0),\ -y+V(0)) \end{bmatrix} = -\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \begin{pmatrix} u \\ y \end{pmatrix}. \tag{13}$$

Then R_G *and* R_K *are well-defined, coprime stable kernel representations for* G *and* K, *respectively.*

Remark 2.12 The minus sign in (13) may seem counter-intuitive, however it assures that in the linear case it is equivalent to

$$\begin{bmatrix} \tilde{V} & \tilde{U} \\ \tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1}$$

Remark 2.13 The expressions for R_G and R_K may be explicitly stated as follows

$$R_G(u,y) = \begin{bmatrix} 0 & -I \end{bmatrix} \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \begin{pmatrix} u - U(0) \\ -y + V(0) \end{pmatrix}$$

$$R_K(y,u) = \begin{bmatrix} -I & 0 \end{bmatrix} \begin{bmatrix} M & -U \\ -N & V \end{bmatrix}^{-1} \begin{pmatrix} -u + M(0) \\ y - N(0) \end{pmatrix}$$

Note that only the well-posedness of $\{G, K\}$ is necessary for the existence and well-definedness of R_G and R_K. Coprimeness and stability follow from the stability of $\{G, K\}$.

Remark 2.14 R_G and R_K are well-defined kernel representations. The expressions for G_{z_G}, resp. K_{z_K}, found by solving $R_G(u,y) = z_G$, resp. $R_K(u,y) = z_K$, are given by

$$G_{z_G} = G(u - U(0) + U(-z_G)) + V(0) - V(-z_G) \tag{14}$$

$$K_{z_K} = K(y - N(0) + N(-z_K)) + M(0) - M(-z_K) \tag{15}$$

The main results of the paper follow from a careful study of the properties of the closed loop system $\{G_{z_G}, K_{z_K}\}$, and the closed-loop kernel operator R_{GK}. We believe that these results are of independent interest, and so present them in the following section.

3. Properties of the Closed Loop $\{G, K\}$

Firstly the properties of the feedback operator R_{GK} are developed. The following result summarizes the relationships which hold between the signals z_G, z_K, y and u.

Lemma 3.1 *Consider the operators* $G : \mathcal{U} \to \mathcal{Y}$, $K : \mathcal{Y} \to \mathcal{U}$ *with well-defined kernel representations* $R_G : \mathcal{U} \times \mathcal{Y} \to \mathcal{Z}_G$, $R_K : \mathcal{Y} \times \mathcal{U} \to \mathcal{Z}_K$. *Suppose that* $\{G, K\}$ *is null-well-posed, then* R_{GK} *is invertible, and there exists an invertible operator* $T_{GK} : \mathcal{Y} \times \mathcal{Z}_K \to \mathcal{Z}_G \times \mathcal{U}$ *such that*

$$T_{GK}\begin{pmatrix} z_K \\ y \end{pmatrix} = \begin{pmatrix} u \\ z_G \end{pmatrix} \text{ iff } R_{GK}\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} z_K \\ z_G \end{pmatrix} \tag{16}$$

Furthermore, if R_G *and* R_K *are stable and* $\{G, K\}$ *is null-stable,* $R^{-1}_{\{G,K\}}$ *is stable. The operator* T_{GK} *is not stable, however the following properties hold for* $\begin{pmatrix} u \\ z_G \end{pmatrix} = T_{GK}\begin{pmatrix} z_K \\ y \end{pmatrix}$.

$$z_G,\ u \text{ stable} \ \Rightarrow\ z_K \text{ stable} \Leftrightarrow y \text{ stable} \tag{17}$$

$$z_K,\ y \text{ stable} \ \Rightarrow\ z_G \text{ stable} \Leftrightarrow u \text{ stable} \tag{18}$$

Proof. If $\{G, K\}$ is null-well-posed, then by definition R_{GK} is invertible. Consider now that y and z_K are given. Then by well-definedness of R_K, there

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{bmatrix} M & -U \\ -N & V \end{bmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{bmatrix} M & -U \\ -N & V \end{bmatrix} \begin{pmatrix} -z_K \\ -z_G \end{pmatrix} \right) - \begin{bmatrix} U(-z_G) - U(0) \\ N(-z_K) - N(0) \end{bmatrix} \tag{21}$$

exists a unique u satisfying $R_K(y,u) = z_K$. Defining $z_G = R_G(y,u)$, gives the output pair of T_{GK}. Explicitly:

$$T_{GK} \begin{pmatrix} z_K \\ y \end{pmatrix} = \begin{pmatrix} R_K^{\#}(y, z_K) \\ R_G(y, R_K^{\#}(y, z_K)) \end{pmatrix}$$

Invertibility of R_{GK} implies that this is the unique pair satisfying this identity, and thus T_{GK} is invertible on its range. Consider now that u and z_G are given. The inverse operator may be constructed as

$$T^{-1}_{\{G,K\}} \begin{pmatrix} u \\ z_G \end{pmatrix} = \begin{pmatrix} R_G^{\#}(u, z_G) \\ R_G(u, R_K^{\#}(u, z_G)) \end{pmatrix}$$

and (16) holds.

If $\{G, K\}$ is null-stable, then by definition R_{GK} is stable. The properties of T_{GK} follow from the unimodularity of R_{GK}. ∎

A careful consideration of equations (14) and (15) leads to the following result, which is crucial to the development of the main results of the paper.

Theorem 3.2
Let $\{G, K\}$ be a well-posed and stable feedback system and suppose that G and K have right coprime factorizations, $G = NM^{-1}$, $K = UV^{-1}$, and define G_{z_G} and K_{z_K} as in (14), (15). Then for all z_G, z_K the feedback system $\{G_{z_G}, K_{z_K}\}$ is well-posed, and is internally stable if and only if z_G, z_K are stable.

Proof. The proof follows once it is recognized that the feedback loop $\{G_{z_G}, K_{z_K}\}$ is equivalent to that of $\{G, K\}$, where the external inputs have been modified.

The problem of proving well-posedness of $\{G_{z_G}, K_{z_K}\}$ is equivalent to that of solving

$$\begin{aligned} u &= w_1 + K_{z_K} y \\ y &= w_2 + G_{z_G} u \end{aligned} \tag{19}$$

for u and y. Defining the signals

$$\begin{aligned} u^\star &= u + U(-z_G) - U(0) \\ y^\star &= y + N(-z_K) - N(0) \\ w_1^\star &= w_1 - M(-z_K) + M(0) + U(z_G) - U(0) \\ w_2^\star &= w_2 - V(-z_G) + V(0) + N(-z_K) - N(0) \end{aligned}$$

and applying (14), (15), it may be seen that solving (19) is equivalent to solving

$$\begin{aligned} u^\star &= w_1^\star + K y^\star \\ y^\star &= w_2^\star + G u^\star \end{aligned} \tag{20}$$

By well-posedness of $\{G, K\}$ the solution is given by

$$\begin{pmatrix} u^\star \\ y^\star \end{pmatrix} = \begin{bmatrix} I & -K \\ -G & I \end{bmatrix}^{-1} \begin{pmatrix} w_1^\star \\ w_2^\star \end{pmatrix}$$

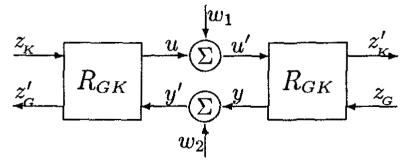

Figure 2: An equivalent representation of $\{G_{z_G}, K_{z_K}\}$

Thus u and y are given by (21) and we may conclude that for all z_G, z_K the system is well-posed.

Stability may be proven by observing that all operators on the right hand side of (21) are stable operators. Thus if z_G, z_K are stable, u, y are stable if and only if w_1 and w_2 are stable. Thus the system is internally stable.

Conversely, if z_G or z_K is unstable, then due to unimodularity of the operator $\begin{bmatrix} M & -U \\ -N & V \end{bmatrix}$, the signal within the brackets in (21) must be unstable for stable w_1, w_2, and thus u or y must be unstable. ∎

Note that this is a much stronger result than null-stability or null-well-posedness of the feedback system. Setting $w_1 = 0, w_2 = 0$ yields the following result.

Corollary 3.3 *Let $\{G, K\}$ be a well-posed feedback system and suppose that G and K have right coprime factorizations, as in (3), (4), and define R_G and R_K as in (13). Then the system $\{G, K\}$ is null-well-posed and if $\{G, K\}$ is internally stable it is null-stable.*

Thus, given $\{G, K\}$ well-posed and stable, $\{S, Q\}$ well-posed and stable, and $w_1 = 0$, $w_2 = 0$, we may construct a Youla parameterization result based on the *skrs* of G, K, S and Q, as in [9]. In this paper we are interested in proving the same result for the case that w_1, w_2 are non-zero.

Theorem 3.2 may be applied to analyze the system depicted in Figure 2. These results are summarized in Lemma 3.4. Before stating the lemma, define

$$e = \begin{pmatrix} y \\ u \end{pmatrix}, \; w_{12} = \begin{pmatrix} w_2 \\ w_1 \end{pmatrix}.$$

Lemma 3.4 *Consider operators $G : \mathcal{U} \to \mathcal{Y}$ and $K : \mathcal{Y} \to \mathcal{U}$ with right coprime factorizations, as in (3), (4), such that $\{G, K\}$ is well-posed. Then the feedback system $\{G_{z_G}, K_{z_K}\}$ is well-posed, and given any two of the signals $\{z_{GK}, z'_{GK}, w_{12}, e, e'\}$, the remaining three signals are uniquely determined.*

Further, if $\{G, K\}$ is stable, then the feedback system $\{G_{z_G}, K_{z_K}\}$ is well-posed and stable. The stability of the system may be tested as follows: Choose two signals, a, b from $\{z_{GK}, z'_{GK}, w_{12}, e, e'\}$. Then if $\{a,b\} \notin \{\{z_{GK}, e'\}, \{z'_{GK}, e\}\}$, stability of a and b is sufficient for stability of all signals in the loop. Otherwise, if a, b and one of the remaining (uniquely determined) signals are stable, the remainder are also guaranteed to be stable.

Proof. Given well-posedness of $\{G, K\}$, the operators R_{GK} and T_{GK} of Lemma 3.1 exist and are invertible. Application of R_{GK}, T_{GK}, and Theorem 3.2 then gives methods of determining the remaining signals, given an initial pair. The only cases which are not straightforward are when considering the pairs $\{z'_{GK}, w_{12}\}$ and $\{z'_{GK}, e'\}$. Note, however that the system is equivalent to that constructed by exchanging the roles of z_{GK} and z'_{GK}, e' and e, and setting $w_{12} := -w_{12}$. Existence and uniqueness of the remaining signals then follows from well-posedness of $\{G_{z_G}, K_{z_K}\}$.

Given that $\{G, K\}$ is stable, the operator R_{GK} is unimodular, and the closed-loop system operator for $\{G_{z_G}, K_{z_K}\}$ is also stable. This guarantees stability in all cases where the existence of the remaining signals was not determined by use of the operator T_{GK}. The operator T_{GK} was only used in the cases where the pairs chosen were $\{z_{GK}, e'\}$ or $\{z'_{GK}, e\}\}$. In either of these cases, stability of a third signal allows unimodularity of R_{GK} or internal stability of $\{G_{z_G}, K_{z_K}\}$ to be applied. ■

4. Main Results

The main result of this paper is a generalization of Theorems 3.1 and 3.4 of [9] to the case that the loop is disturbed by external influences. Before stating this result it is necessary to explain how the classes of plants and controllers are parameterized.

Given $\{G, K\}$ and $\{S, Q\}$, where $S : \mathcal{Z}_K \to \mathcal{Z}_G$ and $Q : \mathcal{Z}_G \to \mathcal{Z}_K$, and all operators have kernel representations, and the systems are null-well-posed. Then the systems G_S and K_Q are defined via their stable kernel representations R_{G_S} and R_{K_Q}

$$\begin{array}{rcrcl} R_{G_S} & : & \mathcal{U} \times \mathcal{Y} & \to & \mathcal{Z}_S \\ & & (u,\ y) & \mapsto & R_S(R_G(u,y), R_K(y,u)), \end{array} \tag{22}$$

$$\begin{array}{rcrcl} R_{K_Q} & : & \mathcal{Y} \times \mathcal{U} & \to & \mathcal{Z}_Q \\ & & (y,\ u) & \mapsto & R_Q(R_K(y,u), R_G(u,y)). \end{array} \tag{23}$$

G_S, resp. K_Q, is well-defined if R_S, R_Q, is a well-defined kernel representation for S, respectively Q.

The system kernel representation of $\{G_S, K_Q\}$ is thus given by:

$$R_{G_S K_Q} = R_{SQ} R_{GK} \tag{24}$$

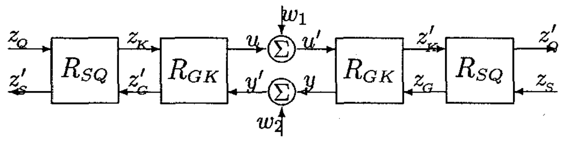

Figure 3: An equivalent representation of $\{G_S, K_Q\}$

Dually, given a system $\{G^\star, K^\star\}$ with kernel representations that is null-well-posed, we may define the operators $S^\star$ and $Q^\star$ which parameterize these via the system kernel representations

$$R_{S^\star Q^\star} = R_{G^\star K^\star} R^{-1}_{\{G,K\}} \tag{25}$$

The main result of the paper may now be stated.

Theorem 4.1
Consider operators $G : \mathcal{U} \to \mathcal{Y}$ and $K : \mathcal{Y} \to \mathcal{U}$ with right coprime factorizations, as in (3), (4), such that the feedback system $\{G, K\}$ is well-posed and stable. Kernel representations giving a null-well-posed and null-stable system are then given by (13). Further, consider the operators $S : \mathcal{Z}_K \to \mathcal{Z}_G$ and $Q : \mathcal{Z}_G \to \mathcal{Z}_K$, which also have right coprime factorizations, and construct the operators $G_S : \mathcal{U} \to \mathcal{Y}$ and $K_Q : \mathcal{Y} \to \mathcal{U}$ as above.

Then the following results hold.

1. *$\{S, Q\}$ is well-posed only if $\{G_S, K_Q\}$ is well-posed.*
2. *$\{S, Q\}$ is internally stable iff $\{G_S, K_Q\}$ is internally stable.*

Furthermore, consider operators $G^\star : \mathcal{U} \to \mathcal{Y}$ and $K^\star : \mathcal{Y} \to \mathcal{U}$ with right coprime factorizations. Then there exist $S^\star$ and $Q^\star$ such that $G^\star = G_{S^\star}$, $K^\star = K_{Q^\star}$, and the following results hold.

1. *$\{S^\star, Q^\star\}$ is well-posed if $\{G^\star, K^\star\}$ is well-posed.*
2. *$\{S^\star, Q^\star\}$ is internally stable iff $\{G^\star, K^\star\}$ is internally stable.*

Remark 4.2 To show that $\{S, Q\}$ is well-posed if $\{G_S, K_Q\}$ is well-posed requires more information about the map R_{GK} than we have currently assumed.

Proof. The proof is constructed by considering the more general system depicted in Figure 3. Denote

$$w_{34} = \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} z'_K - z_K \\ z'_G - z_G \end{pmatrix}$$

Well-posedness is proven by proving the contrapositive. First z_{SQ} is chosen.

Now, suppose $\{S, Q\}$ is not well-posed, then by Lemma 3.4 there exists a w_{34} such that the closed loop equations have no solutions, *i.e.* there exists no z_{GK} which is a solution to the closed loop equations.

However, by well-posedness of $\{G, K\}$ there exists a w_{12}, z_{GK} such that w_{34} is realized. Thus $\{G_S, K_Q\}$ is not well-posed.

Thus $\{G_S, K_Q\}$ is not well-posed if $\{S, Q\}$ is not well-posed, taking the contrapositive, and choosing the signal $z_{SQ} = 0$ proves the first statement in the theorem.

We now prove the second statement. Assume z_{SQ} is stable. Note that by well-posedness of the system, all signals exist.

Now note that if $\{G, K\}$ and $\{S, Q\}$ are stable the operators R_{GK} and R_{SQ} are unimodular, so that z_{SQ} are z'_{SQ} are stable iff z_{GK} are z'_{GK} are stable, iff u, y, u', y' are stable.

Now suppose that $\{G_S, K_Q\}$ is unstable. Then there exists a stable w_{12} which destabilizes the system, in that one of the other signals is unstable. However this violates the unimodularity of R_{GK} and R_{SQ}, so $\{G_S, K_Q\}$ is stable.

Conversely, suppose that $\{S, Q\}$ is unstable. Then there exists a stable w_{34} which destabilizes the system. By stability of $\{G, K\}$ this signal may be realized by a stable w_{12}, which proves that $\{G_S, K_Q\}$ is unstable.

Thus the second statement of the proof has been proven.

The remainder of the theorem is proved in a similar fashion, once the operator $\{S^\star, Q^\star\}$ has been constructed. The identities $G^\star = G_{S^\star}$, $K^\star = K_{Q^\star}$ hold by definition. ∎

Remark 4.3 Strictly speaking this is not the Youla parameterization. It is, however a more general result which allows construction of the Youla parameterization, when S or Q is taken to be zero. The construction of these operators has been done for the noise free case in [9, 12]. For reasons of space, they are not repeated here.

5. Conclusion

In this paper we have shown how the Youla parameterization may be constructed for stable nonlinear feedback systems with right coprime factorizations. This presents a quite complete nonlinear version of the linear results in that state space operators may be factorized via either the right factorization approaches of Verma [18] or Sontag [14], or the left-factorization techniques of Scherpen [13], Sontag [15] or in [9], and then the results of this paper may be applied to parameterize the class of stable feedback pairs.

There is still some work to be done in the case that one has a feedback pair which is stable for a subset of the stable w_{12}. Additionally more information about the forms of the nonlinearities would allow more explicit expressions for the relationships between w_{12} and w_{34}, and would allow more easily tested state-space characterizations of stability.

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